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# On the limit Gibbs states of the spherical model

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**Abstract.** The effect of a class of Hamiltonian perturbations, vanishing in the thermodynamic limit, on the limit Gibbs states of the spherical and mean spherical models is studied. The perturbation term is taken in the form of interaction energy with uniform magnetic field of strength $h_0N^{-\alpha}$, where $h_0 \in \mathbb{R}^1$ and $\alpha > 0$ are parameters, and $N$ is the number of particles. For fixed temperatures below the critical temperature, in the absence of constant external magnetic fields and at $\alpha = 1$ we obtain convex sets of different mixed Gibbs states parametrised by $h_0$. A natural one-parameter generalisation of the Kac-Thompson transformation kernel which relates the states of the mean spherical model to the states of the spherical model is found. When $0 < \alpha < 1$ and $h_0 \neq 0$, or $\alpha = 1$ and $h_0 = \pm\infty$, this kernel becomes a $\delta$ function even below the critical temperature; then the states in both ensembles coincide with each other and with one of the two (depending on the sign of $h_0$) extreme points. The case of $\alpha > 1$ is found to lead to the well known results corresponding to the absence of perturbation ($h_0 = 0$).

## 1. Introduction

This work was inspired by the wish to better understand the discrepancy between the statistical properties of the mean spherical and spherical models which occurs below the critical temperature in the zero-field case. It was first noticed to take place for some average values by Lewis and Wannier (1953), next pointed out for the one-particle probability distribution by Lax (1955) and studied in further generality and detail by Yan and Wannier (1965) and Kac and Thompson (1977). Yan and Wannier (1965) concluded that the discrepancy was not between two different forms of statistics, e.g. between microcanonical and canonical ensembles, but rather within one form of statistics, e.g., the one-particle distribution in the mean spherical model drastically depends on the order of the zero-field limit and the thermodynamic limit. We believe this fact to indicate that the zero-field low-temperature state of the system is unstable and essentially depends on the details of the way the thermodynamic limit is taken.

Here we follow, as well as the authors cited above, the approach to statistical mechanics of infinite systems based on the study of the thermodynamic limit of finite systems described by appropriate Gibbs ensembles. Extending the idea of a recent work by Brankov *et al* (1986), we study the effect of a class of Hamiltonian perturbations on the limit Gibbs states of both the spherical and mean spherical models. The perturbation term is chosen in the form of interaction energy with uniform magnetic field of strength $h_0N^{-\alpha}$, where $h_0 \in \mathbb{R}^1$ and $\alpha > 0$ are parameters, and $N$ is the number of particles in the system. Thus we start with the Hamiltonian

$$\mathscr{H}_N(\boldsymbol{\sigma} | h_0, \alpha) = -\frac{1}{2}\sum_{i,j=1}^{N} T_{ij}\sigma_i\sigma_j - h_0N^{-\alpha}\sum_{i=1}^{N}\sigma_i \qquad \boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_N\} \tag{1.1}$$

where $\sigma_i \in \mathbb{R}^1$ is the dynamical variable associated with the site $i = 1, \ldots, N$ of a $d$-dimensional regular lattice embedded in a torus (i.e. with periodic boundary conditions). The coupling $T_{ij}$ is assumed isotropic and translation invariant. Since the case of a fixed non-zero external magnetic field is trivial from the viewpoint of our investigation, we confine ourselves to the case when the perturbation is included in the zero-field Hamiltonian.

As has been generally accepted after the work of Lewis and Wannier (1952), we consider the spherical and mean spherical models as one statistical mechanical system, described by the Hamiltonian (1.1), in two different Gibbs ensembles. The spherical model of Berlin and Kac (1952) corresponds to the system (1.1) in an ensemble microcanonical with respect to the observable $\Sigma_i \sigma_i^2$, i.e. we fix

$$\frac{1}{N}\sum_{i=1}^{N} \sigma_i^2 = \xi. \tag{1.2}$$

For brevity we will refer to the model obeying condition (1.2) as the spherical model of (normalised) radius $\xi^{1/2}$.

The corresponding grand canonical ensemble leads to the Gaussian model with partition function given by

$$Q_N^{\mathrm{G}}(\beta, s \,|\, h_0, \alpha) = \int \ldots \int \mathrm{d}\sigma_1, \ldots, \mathrm{d}\sigma_N \exp\left(-s \sum \sigma_i^2\right) \exp(-\beta \mathcal{H}_N(\boldsymbol{\sigma} \,|\, h_0, \alpha)). \tag{1.3}$$

The condition for thermodynamic equivalence of the two ensembles leads to the following equation for the activity $\mathrm{e}^{-s}$ in the Gaussian model:

$$-\frac{\partial}{\partial s} \ln Q_N^{\mathrm{G}}(\beta, s \,|\, h_0, \alpha) = N\xi. \tag{1.4}$$

The Gaussian model with activity obeying equation (1.4) is in fact the mean spherical model (of mean normalised radius $\xi^{1/2}$) introduced by Lewis and Wannier (1952).

By repeating the derivation of Yan and Wannier (1965) in the slightly more general case of spherical models of normalised radius $\xi^{1/2}$ for the zero-field one-particle probability distribution

$$p_1(x \,|\, \beta, \xi)\, \mathrm{d}x = \lim_{N\to\infty} \operatorname{Prob}\{x \leqslant \sigma_1 < x + \mathrm{d}x \,|\, \beta, \xi\} \tag{1.5}$$

where the probability is calculated in the appropriate ensemble at inverse temperature $\beta$ and field $h_0 = 0$, one obtains, in the mean spherical model

$$p_1^{\mathrm{ms}}(x \,|\, \beta, \xi) = (2\pi\xi)^{-1/2} \exp(-x^2/2\xi) \tag{1.6}$$

and in the spherical model

$$p_1^{\mathrm{s}}(x \,|\, \beta, \xi) = \begin{cases} (2\pi\xi)^{-1/2} \exp(-x^2/2\xi) & \beta < \beta_{\mathrm{c}}/\xi \\ (2\pi\beta_{\mathrm{c}}/\beta)^{-1/2} \dfrac{1}{2}\left[\exp\left(-\dfrac{[x-(\xi-\beta_{\mathrm{c}}/\beta)^{1/2}]}{2\beta_{\mathrm{c}}/\beta}\right)^2\right. & \\ \quad \left. + \exp\left(-\dfrac{[x+(\xi+\beta_{\mathrm{c}}/\beta)^{1/2}]}{2\beta_{\mathrm{c}}/\beta}\right)^2\right] & \beta > \beta_{\mathrm{c}}/\xi. \end{cases} \tag{1.7}$$

Under the same conditions, the Kac–Thompson transformation kernel (see Kac and Thompson 1977)

$$K(\xi \,|\, \beta) = \lim_{N\to\infty} \operatorname{Prob}\left\{\xi \leqslant \frac{1}{N}\sum_{i=1}^{N} \sigma_i^2 < \xi + \mathrm{d}\xi \,|\, \beta, 1\right\} \tag{1.8}$$

where the probability is calculated in a sequence of mean spherical models of fixed mean radius $\xi^{1/2}=1$ and increasing number of particles $N$, takes the form

$$K(\xi|\beta)=\begin{cases}\delta(\xi-1) & \beta<\beta_c\\ [2\pi(\xi-\beta_c/\beta)(1-\beta_c/\beta)]^{-1/2}\exp\left(-\dfrac{\xi-\beta_c/\beta}{2(1-\beta_c/\beta)}\right) & \beta>\beta_c,\ \xi>\beta_c/\beta\\ 0 & \beta>\beta_c,\ \xi<\beta_c/\beta.\end{cases} \tag{1.9}$$

Kac and Thompson (1977) have shown that for a certain class of observables (functions $f(\boldsymbol{\sigma})$ on the phase space of finite system) the average values in the limit Gibbs states corresponding to the two ensembles are related by the equation

$$\langle f(\boldsymbol{\sigma})\rangle^{\mathrm{ms}}(\beta,1)=\int_0^\infty \langle f(\boldsymbol{\sigma})\rangle^{\mathrm{s}}(\beta,\xi)K(\xi|\beta)\,\mathrm{d}\xi \tag{1.10}$$

where $\langle\ \rangle^{\mathrm{ms,s}}(\beta,\xi)$ is defined as the thermodynamic limit of the corresponding finite system ensemble average.

We will use kernel (1.8) rather than average values to relate probabilities of events in the limit Gibbs states. The argument is based on the consideration of finite system ensembles, for which the probability of some event in the grand canonical ensemble can be expressed as a product of the conditional probability of that event in the microcanonical ensemble at a given radius (condition (1.2)) and the probability of finding the given radius, integrated over all values of the radius. Taking next the thermodynamic limit $N\to\infty$ in the sense of weak convergence of distributions of finite-dimensional random variables $\eta^{(N)}=\{\eta_1^{(N)},\ldots,\eta_n^{(N)}\}$ on the phase space of $N$-particle systems to limit distributions ascribed to some limit random variable $\eta=\{\eta_1,\ldots,\eta_n\}$, assuming the existence and interchangeability of the limits involved as well as the existence of limit probability densities:

$$p_\eta(x_1,\ldots,x_n|\beta,\xi)=\lim_{N\to\infty}\mathrm{Prob}\{x_1\leqslant\eta_1^{(N)}<x_1+\mathrm{d}x_1,\ldots,x_n\leqslant\eta_n^{(N)}<x_n+\mathrm{d}x_n|\beta,\xi\} \tag{1.11}$$

we get a more general form of relation (1.10):

$$p_\eta^{\mathrm{ms}}(x_1,\ldots,x_n|\beta,1)=\int_0^\infty p_\eta^{\mathrm{s}}(x_1,\ldots,x_n|\beta,\xi)K(\xi|\beta)\,\mathrm{d}\xi. \tag{1.12}$$

The formal way of obtaining (1.12) is to set in (1.10) for any fixed integer $n$:

$$f(\boldsymbol{\sigma})=\exp\left(\mathrm{i}\sum_{j=1}^{n}t_j\sigma_j\right). \tag{1.13}$$

Then the average values in (1.10) are meaningful for systems of $N\geqslant n$ particles and are just the characteristic functions of the $n$-dimensional random variable $\eta^{(N)}$ with components $\eta_i^{(N)}=\sigma_i$, $i=1,\ldots,n$. The convergence of characteristic functions in the limit $N\to\infty$ is equivalent to weak convergence of distributions. Equation (1.12) holds true when the transformation from the characteristic function to the density of the distribution function can be performed under the integral over $\xi$.

Relation (1.12) can easily be checked for the one-particle densities (1.6) and (1.7) by using kernel (1.9). In this case it follows formally from equation (1.10) by inserting $f(\boldsymbol{\sigma})=\delta(\sigma_1-x)$.

Relation (1.12) can be inverted in some cases and this fact makes it a useful tool for the calculation of probability distributions in the spherical model based on calculations in the grand canonical (mean spherical) ensemble only.

In this paper we solve the following problems. First we study the effect of the perturbation term in the Hamiltonian (1.1) on the probabilistic kernel (1.8). Working entirely in the grand canonical ensemble, we find in § 2 the modified form of this kernel. In §§ 3 and 4 we study two simple but physically important distributions—the single-spin distribution density $p_1(x)$ and the long-range block spin (using the terminology of Schultz *et al* (1964)) density $p_L(x)$, respectively. We find that the perturbation may affect the limit Gibbs states of the mean spherical model—an indication of this is the change in the shape of $p_1^{ms}(x)$ and $p_L^{ms}(x)$. By inverting equation (1.12) we find the modified densities $p_1^{s}(x)$ and $p_L^{s}(x)$ in the microcanonical ensemble; they reflect the changes in the limit Gibbs states of the spherical model. These results are generalised for arbitrary finite-dimensional projections of the limit Gibbs measures in § 5. Under some technical conditions, the infinite-dimensional characteristic function of the stochastic fields is obtained too. Section 6 contains a short discussion of the results.

## 2. Calculation of the transformation kernel

The characteristic function of the random variable $N^{-1}\Sigma_i \sigma_i^2$ in the mean spherical model with Hamiltonian (1.1) and normalised radius $\xi^{1/2}=1$ is (cf Kac and Thompson 1977)

$$\begin{aligned}\varphi_N(\lambda\,|\,\beta;\,h_0,\alpha) &\equiv \left\langle \exp\left(\mathrm{i}\lambda N^{-1}\sum_j \sigma_j^2\right)\right\rangle^{\mathrm{ms}} \\ &= Q_N^{\mathrm{G}}(\beta, s_N - \mathrm{i}\lambda N^{-1}\,|\,h_0,\alpha)/Q_N^{\mathrm{G}}(\beta, s_N\,|\,h_0,\alpha) \qquad (2.1)\end{aligned}$$

where $s_N = s_N(\beta;\,h_0,\alpha)$ is the solution of equation (1.4) for $\xi=1$. The computation of $Q_N^{\mathrm{G}}(\beta, s\,|\,h_0,\alpha)$ can be easily performed by making use of Fourier transformation which diagonalises the quadratic form in the Hamiltonian. Introducing the Fourier coefficients $\hat{T}(q)$ of the coupling $T_{ij}$, where $q$ takes values in the first Brillouin zone of the reciprocal lattice (see, e.g., Joyce 1972), one obtains explicitly

$$Q_N^{\mathrm{G}}(\beta, s\,|\,h_0,\alpha) = \pi^{N/2}\exp\left(\frac{\beta^2 h_0^2}{4N^{2\alpha-1}(s-\frac{1}{2}\beta\hat{T}(0))}\right)\prod_q \frac{1}{(s-\frac{1}{2}\beta\hat{T}(q))^{1/2}}. \qquad (2.2)$$

Hence, setting $z = 2s_N/\beta\hat{T}(0)$, we find from (2.1):

$$\begin{aligned}\varphi_N(\lambda\,|\,\beta;\,h_0,\alpha) &= \exp\left[\frac{\mathrm{i}\lambda h_0^2/\hat{T}^2(0)}{N^{2\alpha}(z-1)^2}\left(1-\frac{2\mathrm{i}\lambda}{N\beta\hat{T}(0)(z-1)}\right)^{-1}\right] \\ &\quad\times\exp\left[-\frac{1}{2}\sum_q \ln\left(1-\frac{2\mathrm{i}\lambda}{N_\beta\hat{T}(0)(z-\hat{T}(q)/\hat{T}(0))}\right)\right]. \qquad (2.3)\end{aligned}$$

Equation (1.4) at $\xi=1$ can be written in the form

$$R_{d,N}(z) + \frac{1}{N(z-1)} + \frac{\beta h_0^2/\hat{T}(0)}{N^{2\alpha}(z-1)^2} = \beta\hat{T}(0) \qquad (2.4)$$

where

$$R_{d,N}(z) = \frac{1}{N}\sum_{q\neq 0}\frac{1}{z-\hat{T}(q)/\hat{T}(0)}. \qquad (2.5)$$

For a hypercubic lattice of $N = N_0^d$ sites and volume $V = L^d$, where $L = N_0 a$, $a$ being the lattice spacing, under periodic boundary conditions one has

$$q_\alpha = 2\pi p_\alpha / L \qquad p_\alpha = 0, \pm 1, \pm 2, \ldots (\text{mod } N_0) \qquad \alpha = 1, \ldots, d. \tag{2.6}$$

Assuming in addition nearest-neighbour interactions only, the explicit form of $\hat{T}(q)$ is

$$\hat{T}(q) = \hat{T}(0)\frac{1}{d}\sum_{\alpha=1}^{d} \cos\frac{2\pi p_\alpha}{N_0}. \tag{2.7}$$

Now, for $d \geq 3$, by using the method of Fisher and Privman (1986) one can easily obtain an estimate of the sum (2.5) uniform in $z \geq 1$:

$$R_{d,N}(z) = R_d(z) + O(N^{-(d-2)/d}) \tag{2.8}$$

where

$$R_d(z) = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} \ldots \int \frac{\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_d}{z - d^{-1}\Sigma_{\alpha=1}^{d} \cos\theta_\alpha}. \tag{2.9}$$

From (2.4) and (2.8) it follows that, for $\beta < \beta_c = R_d(1)/\hat{T}(0)$, the solution $z = z_N(\beta; h_0, \alpha)$ of equation (2.4) converges in the limit $N \to \infty$ to a value $z(\beta) > 1$ which is independent of the perturbation, provided $\alpha > 0$. Therefore, passing to the limit $N \to \infty$ in (2.3) and taking into account that

$$\lim_{N\to\infty} \sum_q \ln\left(1 - \frac{2\mathrm{i}\lambda}{N_\beta \hat{T}(0)(z_N - \hat{T}(q)/\hat{T}(0))}\right) = -\frac{2\mathrm{i}\lambda}{\beta\hat{T}(0)} R_d(z(\beta)) = -2\mathrm{i}\lambda \tag{2.10}$$

one obtains

$$\lim_{N\to\infty} \varphi_N(\lambda \,|\, \beta; h_0, \alpha) = \mathrm{e}^{\mathrm{i}\lambda} \qquad \beta < \beta_c. \tag{2.11}$$

Then the transformation from characteristic function to probability density leads to the known result

$$K(\xi | \beta; h_0, \alpha) = \delta(\xi - 1) \qquad \beta < \beta_c. \tag{2.12}$$

At $\beta > \beta_c$ there are three different cases since the solution of equation (2.4) converges to $z = 1$ as $N \to \infty$ but the rate of convergence depends on the interplay of the second and third terms on the left-hand side of (2.4).

(i) If $\alpha < 1$ the leading asymptotic form of the solution is

$$z_N(\beta; h_0, \alpha) = 1 + |h_0|[\hat{T}(0)N^\alpha(1 - \beta_c/\beta)^{1/2}]^{-1} \qquad 0 < \alpha < 1. \tag{2.13}$$

By inserting this in expression (2.3) and taking the limit $N \to \infty$ one obtains

$$\lim_{N\to\infty} \varphi_N(\lambda \,|\, \beta; h_0, \alpha) = \mathrm{e}^{\mathrm{i}\lambda} \qquad \beta > \beta_c, 0 < \alpha < 1. \tag{2.14}$$

That is, (2.12) holds true even for $\beta > \beta_c$, provided $0 < \alpha < 1$.

(ii) If $\alpha = 1$, the second and third terms on the left-hand side of (2.4) are of the same order. The leading asymptotic form of the solution is

$$z_N(\beta; h_0, \alpha) = 1 + [N\beta\hat{T}(0)a(\beta, h_0)]^{-1} \tag{2.15}$$

where

$$a(\beta, h_0) = \tfrac{1}{2}\beta^{-2}h_0^{-2}\{[1 + 4(1 - \beta_c/\beta)\beta^2 h_0^2]^{1/2} - 1\}. \tag{2.16}$$

Now (2.3) and (2.15) give

$$\lim_{N\to\infty} \varphi_N(\lambda\,|\,\beta;\, h_0, 1)$$

$$= [1-2\mathrm{i}\lambda a(\beta, h_0)]^{-1/2} \exp\left[\mathrm{i}\lambda\left(\frac{\beta_c}{\beta} + \frac{1-\beta_c/\beta - a(\beta, h_0)}{1-2\mathrm{i}\lambda a(\beta, h_0)}\right)\right]. \qquad (2.17)$$

Hence one obtains a non-trivial generalisation of the Kac–Thompson kernel (compare with (1.9)):

$$K(\xi\,|\,\beta;\, h_0, 1) = \tfrac{1}{2}[2\pi(\xi-\beta_c/\beta)\alpha(\beta, h_0)]^{-1/2}$$

$$\times\left[\exp\left(-\frac{1}{2a(\beta, h_0)}[(\xi-\beta_c/\beta)^{1/2} + (1-\beta_c/\beta - a(\beta, h_0))^{1/2}]^2\right)\right.$$

$$\left. + \exp\left(-\frac{1}{2a(\beta, h_0)}[(\xi-\beta_c/\beta)^{1/2} - (1-\beta_c/\beta - a(\beta, h_0))^{1/2}]^2\right)\right] \qquad (2.18)$$

for $\xi > \beta_c/\beta$, $\beta > \beta_c$ and

$$K(\xi\,|\,\beta;\, h_0, 1) = 0 \qquad \xi < \beta_c/\beta \qquad \beta > \beta_c. \qquad (2.19)$$

(iii) If $\alpha > 1$ the third term on the left-hand side of equation (2.4) is negligibly small in comparison with the second one. The leading asymptotic form of the solution is

$$z_N(\beta;\, h_0, \alpha) = 1 + [N\beta\hat{T}(0)(1-\beta_c/\beta)]^{-1}. \qquad (2.20)$$

By inserting (2.20) into (2.3) and passing to the limit $N\to\infty$ one obtains the result of Kac and Thompson (1977);

$$\lim_{N\to\infty} \varphi_N(\lambda\,|\,\beta;\, h_0, \alpha) = [1-2\mathrm{i}\lambda(1-\beta_c/\beta]^{-1/2} \exp(\mathrm{i}\lambda\beta_c/\beta). \qquad (2.21)$$

This characteristic function corresponds to the probability density (1.9) for $\beta > \beta_c$.

We may remark here that cases (i) and (iii) correspond to the limits $|h_0|\to\infty$ and $h_0\to 0$ of case (ii). Expression (2.18) interpolates continuously between the $\delta$ function and the Kac–Thompson kernel (1.9). Therefore without loss of generality in the following two sections we will confine ourselves to the case $\alpha = 1$.

## 3. Single-spin probability density

Starting from the formal expression

$$p_1^s(x\,|\,\beta, \xi;\, h_0) = \lim_{N\to\infty}$$

$$\times\frac{\int\ldots\int \mathrm{d}\sigma_1, \ldots, \mathrm{d}\sigma_N\, \delta(\xi - N^{-1}\Sigma_i\sigma_i^2)\delta(\sigma_1 - x)\exp[-\beta\mathcal{H}(\boldsymbol{\sigma}\,|\,h_0, 1)]}{\int\ldots\int \mathrm{d}\sigma_1, \ldots, \mathrm{d}\sigma_N\, \delta(\xi - N^{-1}\Sigma_i\sigma_i^2)\exp[-\beta\mathcal{H}(\boldsymbol{\sigma}\,|\,h_0, 1)]} \qquad (3.1)$$

one can easily verify that

$$p_1^s(x\,|\,\beta, \xi;\, h_0) = \xi^{-1/2} p_1^s(\xi^{-1/2}x\,|\,\xi\beta, 1;\, \xi^{-1/2}h_0). \qquad (3.2)$$

Taking into account (3.2) one may write

$$p_1^{\mathrm{ms}}(x(\beta_c/\beta)^{1/2}|\beta, 1; h_0(\beta_c/\beta)^{1/2})$$
$$= \int_{\beta_c/\beta}^{\infty} K(\xi|\beta; h_0(\beta_c/\beta)^{1/2}, 1) p_1^{\mathrm{s}}(x(\beta_c/\beta)^{1/2}|\beta, \xi; h_0(\beta_c/\beta)^{1/2})$$
$$= \int_{\beta_c/\beta}^{\infty} K(\xi|\beta; h_0(\beta_c/\beta)^{1/2}, 1) \frac{1}{\sqrt{\xi}} p_1^{\mathrm{s}}(x(\beta_c/\beta\xi)^{1/2}/\beta\xi, 1; h_0(\beta_c/\beta\xi)^{1/2})\, \mathrm{d}\xi$$
$$= (\beta_c/\beta)^{1/2} \int_0^{\infty} K((\beta_c/\beta)(t+1)|\beta; h_0(\beta_c/\beta)^{1/2}, 1) \frac{1}{(t+1)^{1/2}}$$
$$\times p_1^{\mathrm{s}}\left(\frac{x}{(t+1)^{1/2}}\middle| \beta_c(t+1), 1; \frac{h_0}{(t+1)^{1/2}}\right) \mathrm{d}t \tag{3.3}$$

where the substitution $\xi = (\beta_c/\beta)(t+1)$ has been used. Next we set

$$\beta_c[2\beta a(\beta, h_0(\beta_c/\beta)^{1/2})]^{-1} = s \qquad \beta_c h_0 = \mu. \tag{3.4}$$

Solving the above equations for $\beta$ and $h_0$ one finds

$$\beta = \beta_c(4s^2+2s+\mu^2)(4s^2)^{-1} \qquad h_0 = \mu/\beta_c. \tag{3.5}$$

After expressing $\beta$ and $h_0$ in terms of $s$ and $\mu$ with the use of equations (3.5), denoting the left-hand side of (3.3) as a function of the new variables by $\tilde{p}_1^{\mathrm{ms}}(x|s, \mu)$ and making use of the explicit form (2.18) of the kernel $K$, expression (3.3) simplifies to

$$\tilde{p}_1^{\mathrm{ms}}(x|s, \mu) = \left(\frac{4s^2+2s+\mu^2}{4\pi s}\right)^{1/2} \exp(-\mu^2/4s) \int_0^{\infty} \mathrm{e}^{-st} \frac{\cosh(\mu t^{1/2})}{[t(t+1)]^{1/2}}$$
$$\times p_1^{\mathrm{s}}\left(\frac{x}{(t+1)^{1/2}}\middle| \beta_c(t+1), 1; \frac{\mu}{\beta_c(t+1)^{1/2}}\right) \mathrm{d}t. \tag{3.6}$$

The Laplace transformation in equation (3.6) can be inverted to give

$$\frac{\cosh(\mu t^{1/2})}{[2\pi t(t+1)]^{1/2}} p_1^{\mathrm{s}}\left(\frac{x}{(t+1)^{1/2}}\middle| \beta_c(t+1), 1; \frac{\mu}{\beta_c(t+1)^{1/2}}\right)$$
$$= \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \left(\frac{2s}{4s^2+2s+\mu^2}\right)^{1/2} \exp(\mu^2/4s) \tilde{p}_1^{\mathrm{ms}}(x|s, \mu)\, \mathrm{e}^{st}\, \mathrm{d}s. \tag{3.7}$$

A straightforward calculation of the single-spin probability density in the mean spherical model with perturbation parameter $\alpha = 1$ leads to the result

$$p_1^{\mathrm{ms}}(x|\beta, 1; h_0) = \frac{1}{\{2\pi[1-\beta^2 h_0^2 a^2(\beta, h_0)]\}^{1/2}} \exp\left(-\frac{[x-\beta h_0 a(\beta, h_0)]^2}{2[1-\beta^2 h_0^2 a^2(\beta, h_0)]}\right). \tag{3.8}$$

As expected, in the limit $h_0 \to 0$ this reduces to the known zero-field probability density

$$p_1^{\mathrm{ms}}(x|\beta, 1; 0) = \frac{1}{(2\pi)^{1/2}} \exp(-x^2/2). \tag{3.9}$$

In the limit $h_0 \to \pm\infty$ expression (3.8) recovers the known result for the case when the zero-field limit is taken after the thermodynamic limit:

$$p_1^{\mathrm{ms}}(x|\beta, 1; \pm\infty) = \frac{1}{[2\pi(\beta_c/\beta)]^{1/2}} \exp\left(-\frac{[x \mp (1-\beta_c/\beta)^{1/2}]^2}{2\beta_c/\beta}\right). \tag{3.10}$$

From (3.8) with the use of (3.5) we find

$$\tilde{p}_1^{\rm ms}(x\,|\,s,\mu)=\left(\frac{4s^2+2s+\mu^2}{2\pi(4s^2+2s)}\right)^{1/2}\exp\left(-\frac{s(x-\mu/2s)^2}{2s+1}\right) \tag{3.11}$$

which, after insertion in (3.7), carrying out the integration and transforming back to the initial arguments, gives the final result

$$p_1^{\rm s}(x\,|\,\beta,1;h_0)=\gamma(\beta,h_0)p_1^{\rm s+}(x\,|\,\beta,1)+[1-\gamma(\beta,h_0)]p_1^{\rm s-}(x\,|\,\beta,1) \tag{3.12}$$

where

$$p_1^{\rm s\mp}(x\,|\,\beta,1)=\frac{1}{(2\pi\beta_c/\beta)^{1/2}}\exp\left(-\frac{[x\mp(1-\beta_c/\beta)^{1/2}]^2}{2\beta_c/\beta}\right) \tag{3.13}$$

$$\gamma(\beta,h_0)=\{2\cosh[\beta h_0(1-\beta_c/\beta)^{1/2}]\}^{-1}\exp[\beta h_0(1-\beta_c/\beta)^{1/2}]. \tag{3.14}$$

Obviously, all the known results for the single-spin probability density in the spherical model follow from (3.12)-(3.14) in the limit cases $h_0\to 0$ or $h_0\to\pm\infty$.

## 4. Long-range block spin probability density

Here we study the probability distribution of the random variable $N^{-1}\Sigma\,\sigma_i$ in the thermodynamic limit under the action of perturbation with parameter $\alpha=1$.

In a manner analogous to the one employed in § 3, we start from the definition of the probability density

$$p_{\rm L}^{\rm s}(x\,|\,\beta,\xi;h_0)=\lim_{N\to\infty}$$

$$\times\frac{\int\ldots\int {\rm d}\sigma_1,\ldots,{\rm d}\sigma_N\,\delta(\xi-N^{-1}\Sigma_i\sigma_i^2)\delta(N^{-1}\Sigma_i\sigma_i-x)\exp[-\beta\mathcal{H}(\boldsymbol{\sigma}\,|\,h_0,1)]}{\int\ldots\int {\rm d}s_1,\ldots,{\rm d}\sigma_N\,\delta(\xi-N^{-1}\Sigma_i\sigma_i^2)\exp[-\beta\mathcal{H}(\boldsymbol{\sigma}\,|\,h_0,1)]} \tag{4.1}$$

and find the relation

$$p_{\rm L}^{\rm s}(x\,|\,\beta,\xi;h_0)=\xi^{-1/2}p_{\rm L}^{\rm s}(\xi^{-1/2}x\,|\,\xi\beta,1;\xi^{-1/2}h_0). \tag{4.2}$$

Then, repeating the steps described in equations (3.3)-(3.7), we end up with an equation completely analogous to (3.7), with $p_1^{\rm s}$ and $\tilde{p}_1^{\rm ms}$ replaced by $p_{\rm L}^{\rm s}$ and $\tilde{p}_{\rm L}^{\rm ms}$, respectively.

By direct evaluation in the mean spherical model with perturbation parameter $\alpha=1$, we find

$$p_{\rm L}^{\rm ms}(x\,|\,\beta,1;h_0)=\frac{1}{[2\pi a(\beta,h_0)]^{1/2}}\exp\left(-\frac{[x-\beta h_0 a(\beta,h_0)]^2}{2a(\beta,h_0)}\right). \tag{4.3}$$

Hence

$$\tilde{p}_{\rm L}^{\rm ms}(x\,|\,s,\mu)=\left(\frac{4s^2+2s+\mu^2}{4\pi s}\right)^{1/2}\exp[-s(x-\mu/2s)^2] \tag{4.4}$$

which after insertion in the analogue of equation (3.7), carrying out the integration and restoring the original variables, gives the result

$$p_{\rm L}^{\rm s}(x\,|\,\beta,1;h_0)=\gamma(\beta,h_0)\delta(x-(1-\beta_c/\beta)^{1/2})+[1-\gamma(\beta,h_0)]\delta(x+(1-\beta_c/\beta)^{1/2}) \tag{4.5}$$

where $\gamma(\beta,h_0)$ has been defined in (3.14).

From (4.5) it follows that the law of large numbers is not valid in the limit Gibbs state of the spherical model, obtained under the action of perturbation with $\alpha \geq 1$ and $h_0$ finite (we remind the reader that the case of $\alpha > 1$, $h_0$ finite leads to the same result as $\alpha = 1$ and $h_0 = 0$, which in turn is equivalent to the absence of any perturbation). An arbitrary mixture of the two pure phases characterised by the single-spin probability densities (3.13) can be obtained by means of perturbation with $\alpha = 1$ and an adequately chosen finite value of $h_0$. The pure phases, for which the law of large numbers is valid (see, e.g., Ellis and Newman 1978) are reached in the limits $h_0 \to \pm\infty$ if $\alpha = 1$, or, equivalently, under a perturbation with $0 < \alpha < 1$ and $h_0 \neq 0$. The general structure of the limit Gibbs states will be obtained in the next section.

## 5. Characteristic function of the Gibbs field

For any fixed bounded region $\Lambda$ of the lattice $\mathbb{Z}^d$ we consider a sequence of $d$-dimensional hypercubic boxes $B_N$, $B_N \subset \mathbb{Z}^d$, equipped with periodic boundary conditions and such that (i) if $N > N'$ then $B_N \supset B_{N'}$ and (ii) there is some integer $N_\Lambda$ such that $B_N \supset \Lambda$ for all $N \geq N_\Lambda$. By $|B_N| = N$ we denote the number of sites in $B_N$. We assume $N = N_0^d$, where $N_0$ is an odd integer.

Define the characteristic function

$$\varphi^{\text{ms}}_{\Lambda,N}(\{t_j\}|\beta, \xi; h_0, \alpha) = \left\langle \exp\left\{ \mathrm{i} \sum_{j\in\Lambda} t_j\sigma_j \right\} \right\rangle^{\text{ms}}_{B_N} \tag{5.1}$$

where the average value is calculated for a mean spherical model with average normalised radius $\xi^{1/2}$ and Hamiltonian (1.1) for a system in the box $B_N$. By employing Fourier transformation for diagonalisation of the quadratic form in the Hamiltonian and performing afterwards the integrations in the explicit form of (5.1), one easily obtains

$$\varphi^{\text{ms}}_{\Lambda,N}(\{t_j\}|\beta, \xi; h_0, \alpha) = \exp\left( -\frac{1}{2} \sum_{i,j\in\Lambda} A_N(i-j|\beta) t_i t_j + \frac{\mathrm{i}h_0}{N^\alpha \hat{T}(0)(z_N - 1)} \sum_{j\in\Lambda} t_j \right) \tag{5.2}$$

where

$$A_N(i-j|\beta) = \frac{1}{N\beta\hat{T}(0)(z_N-1)} + \frac{1}{\beta\hat{T}(0)N} \sum_{k\neq 0} \frac{\cos k(i-j)}{z_N - \hat{T}(k)/\hat{T}(0)} \tag{5.3}$$

and $z_N = z_N(\beta; h_0, \alpha)$ is the solution of equation (2.4).

We are interested in the thermodynamic limit $B_N \uparrow \mathbb{Z}^d$ $(N \to \infty)$ of the characteristic function (5.2) at fixed $\Lambda$:

$$\varphi^{\text{ms}}_{\Lambda}(\{t_j\}|\beta, 1; h_0, \alpha) = \lim_{N\to\infty} \varphi^{\text{ms}}_{\Lambda,N}(\{t_j\}|\beta, 1; h_0, \alpha). \tag{5.4}$$

If $\beta < \beta_c$ the parameter $z_N(\beta; h_0, \alpha)$ converges in the limit $N \to \infty$ to $z(\beta) > 1$ and the result is

$$\varphi^{\text{ms}}_{\Lambda}(\{t_j\}|\beta, 1; h_0, \alpha) = \exp\left( -\tfrac{1}{2} \sum_{i,j\in\Lambda} A(i-j|\beta) t_i t_j \right) \tag{5.5}$$

where

$$A(i-j|\beta)=\frac{1}{\beta\hat{T}(0)(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int\frac{\cos(\sum_{\alpha=1}^{d}\theta_\alpha(i_\alpha-j_\alpha)/a)\,d\theta_1,\ldots,d\theta_d}{z(\beta)-d^{-1}\sum_{\alpha=1}^{d}\cos\theta_\alpha} \tag{5.6}$$

is the pair correlation function.

For $\beta>\beta_c$ we consider separately the same three cases as in § 2.

(i) If $\alpha<1$ the parameter $z_N$ in (5.2) and (5.3) has to be taken from equation (2.13). In the limit $N\to\infty$ we obtain

$$\varphi_\Lambda^{ms}(\{t_j\}|\beta,1;h_0,\alpha)$$
$$=\exp\left(-\tfrac{1}{2}\sum_{i,j\in\Lambda}A(i-j|\beta)t_it_j+\mathrm{i}\,\mathrm{sgn}(h_0)(1-\beta_c/\beta)^{1/2}\sum_{j\in\Lambda}t_j\right) \tag{5.7}$$

where the correlation function has the form (5.6) with $z(\beta)=1$.

(ii) If $\alpha=1$, the parameter $z_N$ is given by equation (2.15) and now we obtain

$$\varphi_\Lambda^{ms}(\{t_j\}|\beta,1;h_0,1)$$
$$=\exp\left(-\tfrac{1}{2}\sum_{i,j\in\Lambda}[A(i-j|\beta)+a(\beta,h_0)]t_it_j+\mathrm{i}\beta h_0 a(\beta,h_0)\sum_{j\in\Lambda}t_j\right) \tag{5.8}$$

with $z(\beta)=1$ in the function (5.6). The pair correlation function is

$$\langle\sigma_i\sigma_j\rangle^{ms}-\langle\sigma_i\rangle^{ms}\langle\sigma_j\rangle^{ms}=A(i-j|\beta)+a(\beta,h_0). \tag{5.9}$$

(iii) If $\alpha>1$ we take $z_N$ from (2.20) and the result is

$$\varphi_\Lambda^{ms}(\{t_j\}|\beta,1;h_0,\alpha)=\exp\left(-\tfrac{1}{2}\sum_{i,j\in\Lambda}[A(i-j|\beta)+(1-\beta_c/\beta)]t_it_j\right) \tag{5.10}$$

with $z(\beta)=1$ in the definition (5.6) of $A(i-j|\beta)$.

We note that the cases $\alpha<1$ and $\alpha>1$ again correspond to the limits $h_0\to\pm\infty$ and $h_0\to 0$, respectively, of the case $\alpha=1$.

In order to obtain the characteristic function in the spherical model, we first establish the relation

$$\varphi_\Lambda^{s}(\{t_j\}|\beta,\xi;h_0,\alpha)=\varphi_\Lambda^{s}(\{\xi^{1/2}t_j\}|\xi\beta,1;\xi^{-1/2}h_0,\alpha) \tag{5.11}$$

which follows in the limit $N\to\infty$ from the definition of the above functions for any finite region $B_N\subset\mathbb{Z}^d$, $N\geqslant N_\Lambda$. Next we invert the relation

$$\varphi_\Lambda^{ms}(\{t_j\}|\beta,1;h_0,\alpha)=\int_0^\infty K(\xi|\beta;h_0,\alpha)\varphi_\Lambda^{s}(\{t_j\}|\beta,\xi;h_0,\alpha) \tag{5.12}$$

by following the same reasoning as in §§ 3 and 4.

Since in the cases $\beta<\beta_c$ or $\beta>\beta_c$ and $\alpha<1$, $h_0\neq 0$, the kernel $K(\xi|\nu;h_0,\alpha)$ is $\delta(\xi-1)$ (see § 2), then the characteristic functions of the spherical model and the mean spherical model coincide in the form:

$$\varphi_\Lambda^{s}(\{t_j\}|\beta,1;h_0,\alpha)=\varphi_\Lambda^{ms}(\{t_j\}|\beta,1;h_0,\alpha) \tag{5.13}$$
$$(\beta<\beta_c \text{ or } \beta>\beta_c \text{ and } \alpha<1,\ h_0\neq 0).$$

In the case $\alpha = 1$ we start with the integral relation between $\varphi_\Lambda^{\text{ms}}$ and $\varphi_\Lambda^{\text{s}}$, make use of (5.11) and then make the substitution $\xi = (\beta_c/\beta)(t+1)$. The result is

$$\varphi_\Lambda^{\text{ms}}(\{(\beta/\beta_c)^{1/2} t_j\} | \beta, 1; (\beta_c/\beta)^{1/2} h_0, 1)$$
$$= \frac{\beta_c}{\beta} \int_0^\infty K\left(\frac{\beta_c}{\beta}(t+1) | \beta; (\beta_c/\beta)^{1/2} h_0, 1\right)$$
$$\times \varphi_\Lambda^{\text{s}}\left(\{(t+1)^{1/2} t_j\} | \beta_c(t+1), 1; \frac{h_0}{(t+1)^{1/2}}, 1\right) \mathrm{d}t. \tag{5.14}$$

The important point now is that, as follows from (5.8),

$$\varphi_\Lambda^{\text{ms}}(\{(\beta/\beta_c)^{1/2} t_j\} | \beta, 1; (\beta_c/\beta)^{1/2} h_0, 1)$$
$$= \exp\left[-\tfrac{1}{2} \sum_{i,j\in\Lambda} \left(A(i-j|\beta_c) + \frac{\beta}{\beta_c} \tilde{a}(\beta, h_0)\right) t_i t_j + \mathrm{i}\beta h_0 \tilde{a}(\beta, h_0) \sum_{j\in\Lambda} t_j\right] \tag{5.15}$$

where the temperature dependence is contained, apart from the explicit factor $\beta$, only in the function $\tilde{a}(\beta, h_0) = a(\beta, h_0(\beta_c/\beta)^{1/2})$. Therefore, we may use the substitution (3.5) in order to simplify equation (5.14), with the kernel given by (2.18), to the form

$$\tilde{\varphi}_\Lambda^{\text{ms}}(\{t_j\} | s, \mu) = (s/\pi)^{1/2} \exp(-\mu^2/4s) \int_0^\infty \mathrm{e}^{-st} \frac{\cosh(\mu t^{1/2})}{t^{1/2}}$$
$$\times \varphi_\Lambda^{\text{s}}\left(\{(t+1)^{1/2} t_j\} | \beta_c(t+1), 1; \frac{\mu}{\beta_c(t+1)^{1/2}}, 1\right) \mathrm{d}t. \tag{5.16}$$

Here

$$\tilde{\varphi}_\Lambda^{\text{ms}}(\{t_j\} | s, \mu) = \exp\left[-\tfrac{1}{2} \sum_{i,j\in\Lambda} \left(A(i-j|\beta_c) + \frac{1}{2s}\right) t_i t_j + \frac{\mathrm{i}\mu}{2s} \sum_{j\in\Lambda} t_j\right]. \tag{5.17}$$

Finally, inverting the Laplace transformation in (5.16) and restoring the original variables, we obtain

$$\varphi_\Lambda^{\text{s}}(\{t_j\} | \beta, 1; h_0, 1) = \gamma(\beta, h_0) \varphi_\Lambda^{\text{s}+}(\{t_j\} | \beta, 1) + [1 - \gamma(\beta, h_0)] \varphi_\Lambda^{\text{s}-}(\{t_j\} | \beta, 1) \tag{5.18}$$

where

$$\varphi_\Lambda^{\text{s}\pm}(\{t_j\} | \beta, 1) = \exp\left(-\tfrac{1}{2} \sum_{i,j\in\Lambda} A(i-j|\beta) t_i t_j \pm \mathrm{i}(1-\beta_c/\beta)^{1/2} \sum_{j\in\Lambda} t_j\right) \tag{5.19}$$

and $\gamma(\beta, h_0)$ has been defined in (3.14).

Now one may consider the limit $\Lambda \uparrow \mathbb{Z}^d$ under the condition of convergence of the sums in (5.19). In the particular case of $h_0 = 0$ one has $\gamma(\beta, 0) = \frac{1}{2}$ and the results of Molchanov and Sudarev (1975) for $\beta > \beta_c$ follow from (5.18) and (5.19), and for $\beta < \beta_c$ from (5.5) and (5.13).

## 6. Conclusions

From the results obtained in this paper one can make the following main conclusions and conjectures.

(i) Below the critical temperature there are just two pure translation-invariant Gibbs states $\varphi^{s\pm}$ (5.19) for the spherical model. The mixed Gibbs states are convex linear combinations of the two extreme points:

$$\varphi^{s}=\gamma\varphi^{s+}+(1-\gamma)\varphi^{s-} \qquad 0\leqslant\gamma\leqslant 1. \tag{6.1}$$

(ii) Below the critical temperature the mixed Gibbs states of the mean spherical model are of the form

$$\varphi^{ms}=\int_{\beta_c/\beta}^{\infty}\{\gamma(\xi)\varphi^{s+}(\xi)+[1-\gamma(\xi)]\varphi^{s-}(\xi)\}\,d\nu(\xi) \tag{6.2}$$

where $d\nu(\xi)$ is a probability measure on the interval $[\beta_c/\beta,\infty)$, $\varphi^{s\pm}(\xi)$ are the pure Gibbs states of the spherical model of normalised radius $\xi^{1/2}$ and $0\leqslant\gamma(\xi)\leqslant 1$. The measure which describes the mixed states appearing under the action of vanishing uniform perturbation with parameters $\alpha$ and $h_0$ is

$$d\nu(\xi|\beta;h_0,\alpha)=K(\xi|\beta;h_0,\alpha)\,d\xi \tag{6.3}$$

where the explicit expressions for the probability density $K(\xi|\beta;h_0,\alpha)$ have been obtained in § 2.

(iii) In the co-existence region, the pure Gibbs phases of both models can be singled out by the vanishing in the thermodynamic limit perturbation of the Hamiltonian with $0<\alpha<1$.

(iv) As generally expected, vanishing of the pair correlation function $\langle\sigma_i\sigma_j\rangle-\langle\sigma_i\rangle\langle\sigma_j\rangle$ in the limit $|i-j|=\infty$ is equivalent to pureness of the Gibbs state.

(v) Statistical inequivalence of the grand canonical and microcanonical Gibbs ensembles exists only in mixed states; on the set of pure states the Gibbs ensembles are statistically equivalent.

As far as generalisations of these results are concerned, we may note the following.

In the case of ferromagnetic $O(n)$ systems in $d>2$ dimensions, the finite-size scaling analysis of Fisher and Privman (1985) predicts that the leading order behaviour of the magnetisation below the critical temperature $T_c$ is

$$m(H,T;L_j)\approx m_0(T)I_{(1/2)n}(y_V)/I_{(1/2)n-1}(y_V). \tag{6.4}$$

Here $H$ is the external magnetic field, $L_j$, $j=1,\ldots,d$, are the lengths of the edges of the block sample, so that its volume is $V=L_1,\ldots,L_d$, $m_0(T)=(1-\beta_c/\beta)^{1/2}$ is the spontaneous magnetisation, $I_\nu(y)$ are the Bessel functions of imaginary argument and $y_V=\beta m_0 HV$ is the natural dimensionless scaling combination. Hence one may conjecture that in the case of $O(n)$ models exhibiting first-order phase transition, non-trivial mixing of the pure states will take place for $y_V=O(1)$, i.e. for $\alpha=1$ in our notation. This has been shown rigorously in the case of the mean spherical model by Fisher and Privman (1986). Naturally, the scaling form of the magnetisation found in the latter work (their equations (3.5)–(3.7)) coincides with the magnetisation per spin:

$$m=\beta h_0 a(\beta,h_0)=m_0(T)\,Y_0(y_V) \qquad Y_0(y)=2y/[1+(1+4y^2)^{1/2}] \tag{6.5}$$

which folllows from our expressions (4.3) and 2.16) for the mean spherical model, but does not coincide with the corresponding expression for the spherical model of Berlin and Kac (1952):

$$m = m_0(T)[2\gamma(\beta, h_0) - 1] = m_0(T) \tanh y_V \tag{6.6}$$

which follows from our result (4.5), although the qualitative similarity is apparent. For the sake of completeness and independence of the results we have considered the cases of $\alpha < 1$ and $\alpha > 1$ too. The main point here is the proof that not just the magnetisation but the whole probability distribution coincides in the thermodynamic limit with that of the pure phase if $\alpha < 1$.

The rigourous investigation of $O(n)$ mean-field model carried out by Angelescu and Zagrebnov (1985) has confirmed the expectation than its pure limit Gibbs states correspond to points of a sphere in $R^n$ with radius equal to the spontaneous magnetisation (below the critical temperature). In the particular case of vanishing uniform external field it was found that non-trivial mixing of the pure states takes place again for $\alpha = 1$. Of course, the structure of the set of limit Gibbs states and the character of mixing are different from those of the spherical model.

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